EXPLAINING MATHEMATICS

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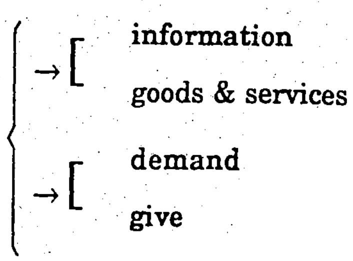
This paper argues, with examples, that as a result of how mathematics is presented, especially in textbooks, explanation is highly implicit. Students are given mathematical problems and are told how to do them, but the whys - the underlying principles and connections - are understated. Thus students are left to infer these from examples, visual representations and manipulations, with the result that mathematics makes little sense to many learners. A strategy for determining at what points explanation breaks down is suggested.

As a way of getting your attention, I'd like to suggest that teaching mathematics and consulting a doctor have a lot in common. From a description of a patient's symptoms and clinical signs, a doctor must:
(a) identify what the problem is;
(b) know how to treat it;
(c) know why this is the best course of action; and these days,
(d) explain to the patient the what, how and why.
Mathematics instruction aims to enable students
(a) to identify what the problem is from linguistic and mathematical symbols;
(b) to know how to solve the problem;
(c) to understand why the solution works; and according to recent curriculum documents,
(d) to explain what they've done and why.
Knowing and explaining **why** a solution works proves problematical for many learners. This is in part because explanation in mathematics teaching is often implicit rather than explicit. Underlying principles, precepts and especially relationships between mathematical phenomena are presented but **not** in so many words, indeed often not in words at all.

Some may say 'so what?'. Mathematics is not language; mathematics is a meaning system distinct from language. I don't agree. Mathematics is as grounded in language as Literature or English. To illustrate what I mean by this, I will show you first an abstraction from my work and then one from mathematics.

<u>Illustration 1</u>:



demand: info	= Question
give: info	= Statement
demand: g & s	= Command
give: g & s	= Offer

This configuration is called a system network. It denotes the fact that speakers have a choice of giving or demanding. That which they give or demand is goods and services or information. So if a speaker demands information, she is asking a Question (e.g. What's this bird on about?) If she gives information, a Statement is being made (e.g. She's prattling on about language). A Command is a demand for goods and services (e.g. Shut up and listen). And should she give goods and services, she's making an Offer (e.g. I'll just move this transparency up and focus it better).

The graphics and terms of the above network are meaningful but their meaning could not be revealed to you *except through language*. The same is true of mathematics, which is represented through graphic and linguistic symbols.
Ilustration 2:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

To explain the meaning of this mathematical representation, language is needed. Mathematics is constituted of language no less than any other domain of knowledge. Without language there is no mathematics.

This latter statement - without language there is no mathematics - is both a truism and a problem. It is a problem because of the way mathematics is presented in teaching texts. Such texts, both spoken and written, make explicit the 'how to' much better than the 'why'. This claim and others made in this address are based on research exploring the language of classroom mathematics conducted on behalf of the Disadvantaged Schools Program, Metropolitan East Region, in Sydney over the last two years.

Mathematics instruction typically includes five phases:
1. Introduction
2. Activity
3. Information Giving
4. Instruction
5. Practice.

The **Introduction** provides an orientation to the topic at hand.
Example 1a:

A girl has made this beautiful patchwork quilt.
It could cover a bed, or hang on a wall. The
amount of space it covers is called its **area**.
(Barry, Booker, Perry and Siemon, 1984:172)

Example 1b:

Not long ago in Australia we used the Imperial system
of measurement. In this system we had no consistent
base for grouping units. For example there were 12
inches in 1 foot, 3 feet in 1 yard and 1760 yards in
1 mile. The system we now use is the International
System of Units known as the SI system. This system,
like our number system, is based on groupings of ten
and powers of ten. This makes caculation much simpler.
(Jones and Couchman, 1982)

The **Activity** phase involves manipulating concrete materials as part of the learning experience. This is most prominent in lower and middle primary school.
Example 2a:

Cover the front of your maths book with envelopes
and then with playing cards. Count how many are
used each time. (Barry, et al, 1984:172)

Example 2b:

Tchr: Now this first stencil that I'm going to give you has
the solids on the left hand side of the sheet and the nets on
the right hand side. So what I want you to do is to try and
imagine. What a net is is what the solid would look like if
you could make it out of cardboard and you unfolded it.
Alright, if you made a cube out of cardboard and if you

could unfolded it and lay it flat what would it look like. Alright, so what I want you to do on that right hand side of the page is have a little bit of an experiment with what the nets of the solids would look like.

(Maths in Society: Year 11, NSW)

These two examples are significant not only for illustrating **Activity,** but also for indicating a major difference in what constitutes *doing* mathematics in primary and secondary school. In the research database on which the paper is based, primary school students actually engaged in drawing, measuring, laying flat, cutting, folding and taping together nets. In the secondary school, the doings were all mental, happening only in the learners' imaginations - "try and imagine", "if we were to unfold that cube, lay it out flat". If our observations are correct, primary school mathematics students are involved in physically manipulating concrete materials and secondary school students in mentally manipulating symbolic representations. Whether there is an appropriate transition from doing mathematics materially in the primary school to doing it mentally and symbolically in secondary school is currently under investigation.

The third phase of mathematics instruction is **Information Giving.**

Information Giving involves introducing technical terms, defining these and providing formulae where appropriate. These comprise the most explicitly stated relevant mathematical concepts.

Example 3a:

This is one centimetre grid paper.

Each square has sides of 1 centimetre long.

Each square covers an area of **1 square centimetre.**

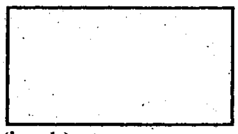
(Barry, et al, 1984:175)

Example 3b:

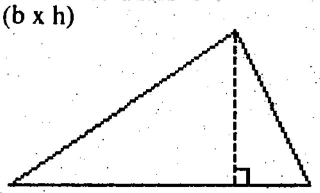
Formulae can be used for finding the area of many plane figures instead of counting squares on graph paper.

1. Square $A = l^2$

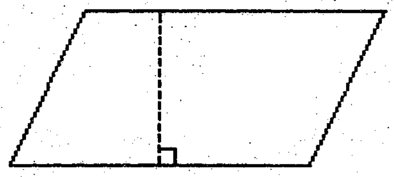
2. Rectangle $A = l \times b$



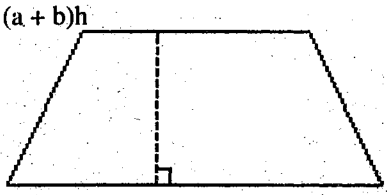
3. Triangle $A = 1/2 (b \times h)$



4. Parallelogram $A = l \times h$



5. Trapezium A = 1/2 (a + b)h



(Lynch, Parr and Keating, 1987:325-26)

As someone not overly gifted in mathematics, I have to accept the above formulae on faith, because they are presented as fact without explanation.

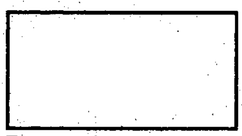
So I may be able to calculate the area of a triangle without knowing why the area of a triangle is half that of a rectangle (any old rectangle?), just as I may be able to invert and multiply fractions to calculate 1/3 - 4/9 without knowing why I'd ever want to divide fractions in the first place, and without knowing why one inverts and multiplies.

The triangle example above is highlighted deliberately, because I've been having trouble with the area of a triangle since starting to analyze the language of classroom mathematics. The first 'explanation' I found was as follows.

Starting with the formula for a rectangle the other shapes
below can be dissected and rearranged to form rectangles.
When this is done the truth of each formula is demonstrated.

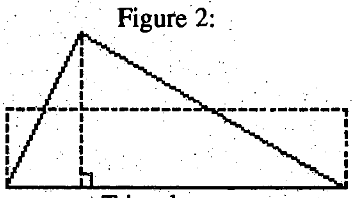
(Jones and Couchman, 1982)

Figure 1:



Rectangle
A = b x h

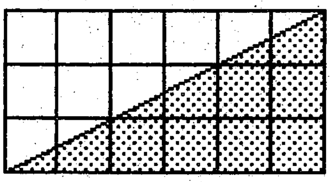
Figure 2:



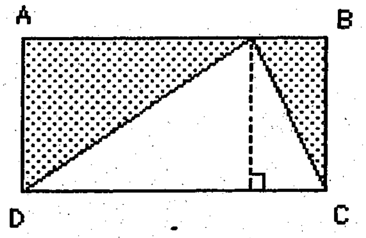
Triangle
A = 1/2 (b x h)

I can see that the first part of this statement is true. But I fail to see how dissecting and rearranging Figure 2 demonstrates the truth of (proves?) the formula. In this instance the explanation is assumed in the visuals provided. This is very common in mathematics instruction. Visual representations carry much of the weight of explanation. Accompanying verbal explanation is apparently thought unnecessary.

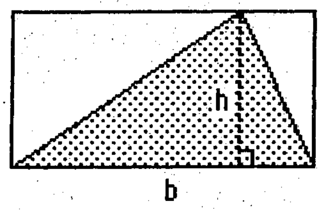
I thought my problems were finally over when I came across the explanation provided in a NSW Year 7 textbook (Jones and Couchman, 1978:101):



What is the area of the rectangle?
What is the area of the triangle?
Is the area of the triangle half the area of the rectangle?

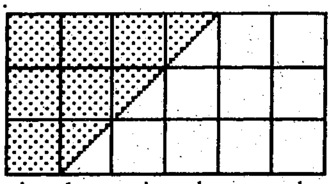


Copy the diagram onto a piece ofpaper and cut out the rectangle ABCD.Cut off the two shaded triangles and fit them on triangle DPC. Do they fit exactly?
Does this mean that the area of triangle DPC is half the area of the rectangle ABCD?

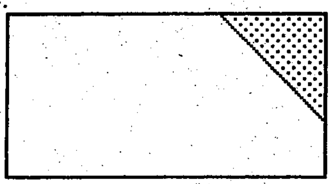


Since the area of a triangle is half of the rectangle containing it, the area of a triangle is given by $A = 1/2 (b \times h)$ where b is the number of units of length in the base and h is the number of units of length in the perpendicular height.

This explanation is convincing as far as it goes. But what if I want to divvy up my rectangle as follows? Is this allowed?
Figure 3:



If every triangle, not just the ones shown above, have a 'rectangle containing it', am I on safe ground if I draw my proof as follows:
Figure 4:

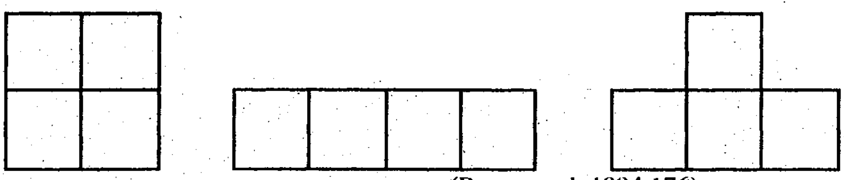


Answers of 'no' to the above two questions suggest that there is a hidden agenda, something that still isn't being explained.

In the fourth phase of mathematics teaching, **Instruction**, sample problems are solved step by step. These steps entail giving the parameters of the problem (quantities); a task - that is, the problem to be solved; and a solution, in which the values specified as parameters are plugged into the formula provided.

Example 4a:

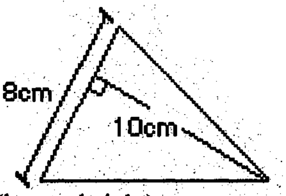
Here are some shapes with an area of $4cm^2$. They have different shapes but they have the same area.



(Barry,,et al, 1984:176)

Example 4b:

Find the area of



Solution: Area = 1/2 (base x height)

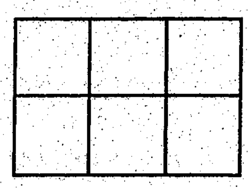
$= 1/2\ (8 \times 10)\ cm^2$

$= 40cm^2$

(Haese, et al, 1983:137)

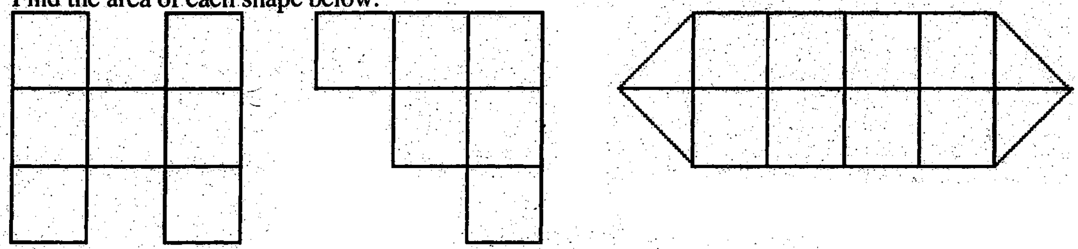
Then students are given **Practice** exercises to do.

Example 5a:



This shape has an area of $6cm^2$

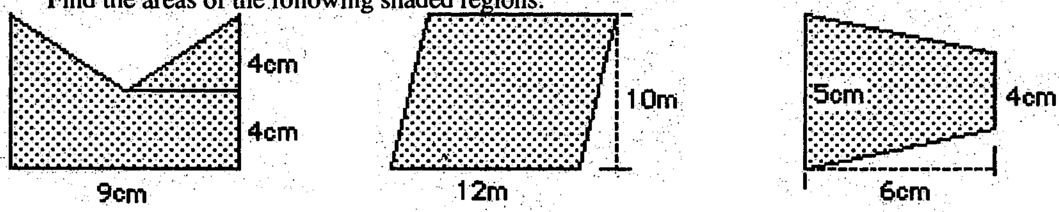
Find the area of each shape below.



(Barry, et al, 1984:177)

Example 5b:

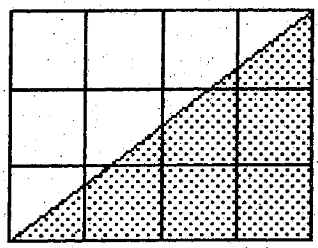
Find the areas of the following shaded regions:



(Haese, et al, 1983:138)

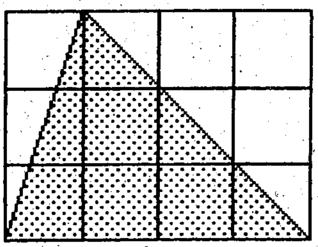
I would like to return briefly to the **Instruction** phase. This phase is usually given the most attention and time/space in mathematics teaching. I see two major problems here for learners. Firstly, explanation at this stage is primarily procedural. Students are shown 'how to' solve given problems (ones already identified for the students), but the connection between the 'how to' and the 'why' is left for the learner to infer. Moreover, universal principlies are taught through particular instances. Students must determine, with little guidance, which bits of an example apply to all instances of the phenomenon unders focus and which apply to the particular instances demonstrated. Consider again the area of a triangle.

Consider the diagrams on the left. Notice that:



1. The base of the triangle is equal to the length of the rectangle.

2. The height of the triangle is equal to the width of the rectangle.



3. The area of the triangle is equal to half the area of the rectangle.

From this we can see that:
The area of a triangle = half the base x the height.
(Hollands and Moses, 1980:50)

The learner can see that in this case the base of the triangle is equal to the length of the rectangle, etc., but is this always so? How is the learner to know? After all, it isn't in Figure 4!!

If the data analyzed are representative, it seems that mathematical principles and their relationships to each other are often not made explicit, are not stated in so many words at any phase. Instead, students are left to infer these for themselves from (1) manipulation of concrete materials in primary schools, (2) diagram/formula pairs as in Figures 1 and 2 above, and/or (3) exemplification using particular instances. What principles, for example, *am* I to understand when confronted with Figures 1 and 2 above? What connection do they have to each other? To me?

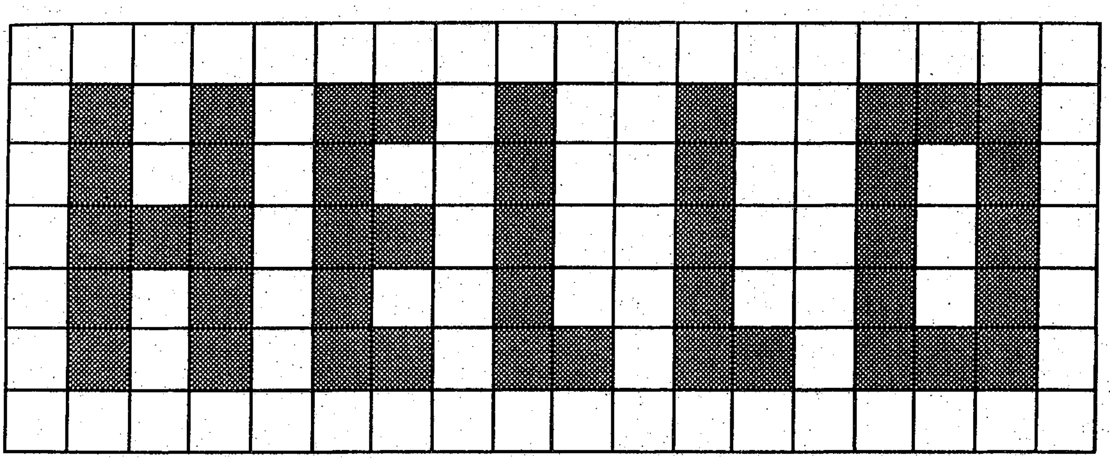
Exemplification, so prominent in the **Instruction** phase, is necessary for showing students how to solve problems, but is not sufficient to ensure understanding of the principles involved. Thus students are often baffled when confronted with word or real life problems. They do not even know what the problem *is* because they are not able to recognise that the new problem is in principle the same as one they've already met. In other words, when basic principles are not understood, because they are not spelled out in so many words, learners cannot generalise from the exemplified instances to new instances. Each problem is seen as novel and unfamiliar.

Mathematics teachers may not be aware of the extent to which learners are required to infer concepts and connections in mathematics. Certainly writers of mathematics textbooks seem unaware of the need to **explicitly** explain. Because mathematicians understand the principles involved, inferencing is automatic and the gaps in teaching texts may not be at all obvious. But if a learner says "I don't get it", chances are some principle has not been adequately explained.

Perhaps explanation implicit in manipulation, diagrams and examples is sufficient; perhaps discursive explanation would be superfluous. To check these possibilities, small groups of NSW Year 4,6,9 and 11 students were given sets of tasks for which the answers were provided. The students were asked to explain why the answers were as they were. The problems were taken from work done earlier by the students so should have been familiar.

Year 4 example:

Perimeter is the distance around the outside of an object or shape.
Surface area is the amount of space something covers.
Pretend your friends don't know anything about perimeter or surface area. Teach them how to complete the table below.



Perimeter	Surface Area
H = 24cm	H = 11cm^2
E =	E =
L =	L =
L =	L =
O = 16cm	O = 12cm^2

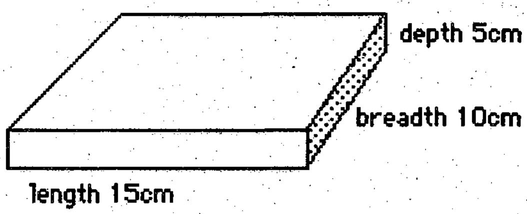
The group of six immediately split into two camps, one of which read and considered the definitions and answers provided, the other which launched straight into deriving formulae for calculating perimeter and surface area. The former subgroup had no difficulty linking the definitions and answers provided, and using these to derive accurate answers for the perimeter and surface areas of letters E and L. The latter group decided that to calculate perimeter, one must add top plus bottom to sides times 2: (t + b) + (s x 2). So their calculation for the perimeter of E was 4 + (5 x 2) = 14, and for L was 3 + (5 x 2) = 13. Notice that this 'formula' will work for O, but not for H, a fact the sub-group overlooked. To calculate surface area the second group decided that the formula was 'down plus across'. Using this formula, the surface area of E was calculated to be 11 (5 down and 6 across) and for L was 7 (5 down and 2 across). One member wasn't happy with this and suggested a formula wasn't needed, that just counting the number of squares coloured in would suffice "cause that tells ya surface area".

Year 6 examples:

The Year 6 group was unable to resolve or explain any of its three tasks.

1. The surface area of this prism is

SA = (150 + 75 + 150 + 75 + 50 + 50)cm^2
= 550cm^2
This is the same as
2(150 + 75 + 50)cm^2
= 550cm^2



See if you can discover a rule or formula for finding the surface area of prisms like this.

2. A swimming pool in the shape of a rectangular prism is 20 metres long, 10 metres wide and 2 metres deep. Explain why it would take 400 cubic metres (400,000 litres) of water to fill the pool.
3. What is the volume of the prism in Question 1? How would you explain the difference between volume and surface area to a Year 4 student?

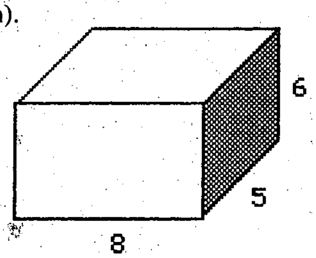
Year 9 examples:

A group of three Year 9 girls had no difficulty explaining the first answer below but took 30 minutes to resolve question 2 and gave up completely on question 3.

1. A plasterer is paid 96 cents per square metre. For plastering two ceilings, one 5.2 metres by 4.5 metres, and the other 6.0 metres by 5.6

metres, he charged the customer $57.00. Explain why the account came to this amount.

2. The surface area of this rectangular prism is $236cm^2$. Explain why the formula for calculating the surface area of a rectangular prism is: S.A. = 2(lb + lh + bh).



Explain why the volume of this same rectangular prism is $240cm^3$. Work out the formula for calculating the volume of a rectangular prism.

3. If the surface area of a rectangle is S.A. = b x h, explain why the surface area of a triangle is S.A. = 1/2 (b x h).

Year 11 examples:

The three tasks comprising the Year 11 set came from a revision unit and caused the students little difficulty. It is interesting to note how similar these tasks are to those set for much younger learners.

1. How would you explain to a Year 7 student why a carpet 4 metres by 3.5 metres laid in a room 6 metres by 4.25 metres leaves an area of floor $11.5m^2$ uncovered?
2. A 5 metre length of fencing timber costs $8.00 and fence posts cost $5.00 each. If 9 lengths of timber are needed to fence a triangular paddock and a fence post is needed for each metre of fence, explain whether the fence timber or the fence posts would cost more and why.
3. A tiler charges $9.60 per square metre for labour. He has charged your granny $56.00 to tile her bathroom floor which measures 2 metres by 2.5 metres. Granny feels that she has been over-charged. How do you help her check?

The students involved in the above tasks varied in their abilities to explain. Several commented that they were used to 'finding the answers' and explaining how they got their answers to teachers, but were not accustomed to being given solved problems to explain. These comments reflect their understandings of how mathematics instruction is presented. Mathematics teaching texts, I have argued, are good at defining problems and explaining how to solve them, but leave the explanation of why much too implicit. Written mathematics especially has become very cryptic and hieroglyphic, requiring sophisticated translation and interpretation skills on the part of both teachers and learners. I suspect it will take a long while for written mathematics to become discursive. Meanwhile, teachers need to be aware that mathematical definitions, manipulations, diagrams, formulae and examples are by no means self-explanatory. They need to be aware of how implicit explanation in mathematics is, and need to make mathematical concepts and connections explicit.

Students can help teachers with this. Engaging students in the kind of explanatory exercises of the kind described above is useful in several ways. Firstly, it is useful diagnostically. Inability to explain suggests that principles and/or connections have not been made sufficiently explicit and so are not understood. Secondly, it provides opportunity for learners to make explicit for themselves what they understand when they do understand. And thirdly, students' explanations are often more intelligible to peers than adult explanations. Students-as-teachers often appreciate better what their peers-as-learners need to hear and how than we do. Thus, while responsibility for knowing what needs to be explained and explaining it falls firstly and primarily to teachers, learners also have a significant role to play in explaining mathematics.

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